A Comparison of Option Pricing Models

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Abstract

Modeling a nonlinear pay off generating instrument is a challenging work. The models that are commonly used for pricing derivative might divided into two main classes; analytical and iterative models. This paper compares the Black-Scholes and binomial tree models.

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JEL classification:

1. Introduction

Modeling a nonlinear pay off generating instrument is a challenging work to handle. If we consider a European option on a stock, what we are trying to do is estimating a conditional expected future value. In other words we need to find out the following question: what would be the expected future value of a stock given that the price is higher than the option's strike price? If we find that value we can easily get the expected value of the option. For the case of the American options the model need to be more complex. For this case, we need to check the path that we reached some future value of the stock, because the buyer of the option might exercise the option at any time until the maturity date.

To solve the problem that summarized above, first we need to model the movement of the stock during the pricing period. The common model for the change of the stock prices is Geometric Brownian Motion. Secondly, the future outcomes of the model might have the same risk. Risk Neutrality assumption provides that. By constructing a portfolio of derivative and share makes possible to have same

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outcome with canceling out the source of the uncertainty.

The models that are commonly used for pricing derivative might divided into two main classes. The first classes is the models that provide analytical formulae to get the risk neutral price under some reasonable assumptions. The Black-Scholes formula is in this group. The formulae that we have to price the derivatives are quite limited. The reason is that we are trying to solve a partial differential equation at the end of the day. But mathematician could manage to solve just some of the partial differential equations; therefore, we are bounded to some limited solutions.

The second classes models provide numerical procedures to price the option. Binomial trees that first suggested by Cox, Ross and Rubenstein, is in this group, because we need to follow an iterative procedure called "backwards induction" to get option price. Monte Carlo simulations are another type of models that belongs to this class. Also finite differencing methods are a type of numerical class.

In this paper, first I will introduce Black-Scholes and Binomial Tree models for option pricing. Second I will introduce the volatility estimation methods I used and calculate some option prices to compare models. Finally I will conclude.

2. Option Pricing Models

2.1. Black-Scholes Model

Black-Scholes formula suggested by Fischer Black and Myron Scholes at 1973. The Black-Scholes (1973) option pricing formula prices European put or call options on a stock that does not pay a dividend or make other distributions. The formula assumes the underlying stock price follows a geometric Brownian motion with constant volatility. The geometrics Brownian motion can be shown as follows;

$$\delta S = \mu S \delta t + \sigma S \delta z$$

where S is the current price of the stock, dS the change in the stock price, μ is the expected rate of return, σ is the volatility and dz is the part follows a Wiener process.

Then using Ito's lemma the price of the option might be shown as;

$$\delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \delta t + \frac{\partial f}{\partial S} \sigma S \delta z$$

where df is the change in the option value.

This differential equation formulates the movement of the option price over time. Now note that the source of the risk in the stock and the option is exactly same (dz). Then if we construct a risk neutral portfolio consisting a short position in the option and a long position in the stock as follows

$$derivative : -1$$
$$share : \frac{\partial f}{\partial S}$$

The pay off function can be shown as;

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change of the pay off function is;

$$\delta \Pi = -\delta f + \frac{\partial f}{\partial S} \delta S$$
$$\delta \Pi = -\left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \delta t - \frac{\partial f}{\partial S} \sigma S \delta z + \frac{\partial f}{\partial S} \left(\mu S \delta t + \sigma S \delta z \right)$$
$$\delta \Pi = -\left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \delta t$$

This portfolio is riskless because we can exclude the source of uncertainty. Since it is riskless it should yield risk free rate otherwise there could be arbitrage opportunities. Then we can write;

$$\delta \Pi = r \Pi \delta t$$
$$\left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \delta t = r \left(f - \frac{\partial f}{\partial S} S \right) \delta t$$

And

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} r S + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = r f$$

This partial differential equation is called Black Scholes differential equation. The solution for this differential equation is the Black-Scholes formulae;

$$c = S_0 N\left(d_1\right) - K e^{-rT} N\left(d_2\right)$$
$$p = K e^{-rT} N\left(-d_2\right) - S_0 N\left(-d_1\right)$$

where

$$d_1 = \frac{\ln(S_0/K) + \left(r + \sigma^2/2\right) T}{\sigma \sqrt{T}}$$
$$d_2 = \frac{\ln(S_0/K) + \left(r + \sigma^2/2\right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}$$

Then the formula needs the current price of the stock S_0 , the stirke price K, risk free short term interest rate r, time to maturity T and volatility of the stock σ($N\left(.\right)$ is the cumulative normal distribution function).

The model is based on a normal distribution of underlying asset returns which is the same thing as saying that the underlying asset prices themselves are log-normally distributed. A lognormal distribution is right skewed distirbution. The lognormal distribution allows for a stock price distribution of between zero and infinity (ie no negative prices) and has an upward bias (representing the fact that a stock price can only drop 100% but can rise by more than 100%).

In practice underlying asset price distributions often depart significantly from the lognormal. For example historical distributions of underlying asset returns often have fatter left and right tails than a normal distribution indicating that dramatic market moves occur with greater frequency than would be predicted by a normal distribution of returns– ie more very high returns and more very low returns.

The main advantage of the Black-Scholes model is speed – it lets you calculate a very large number of option prices in a very short time.

The Black-Scholes model has one major limitation: it cannot be used to accurately price options with an American-style exercise as it only calculates the option price at one point in time – at expiration. It does not consider the steps along the way where there could be the possibility of early exercise of an American option.

As all exchange traded equity options have American-style exercise (ie they can be exercised at any time as opposed to European options which can only be exercised at expiration) this is a significant limitation.

The exception to this is an American call on a non-dividend paying asset. In this case the call is always worth the same as its European equivalent as there is never any advantage in exercising early.

2.2. Binomial Model

The binomial model breaks down the time to expiration into potentially a very large number of time intervals, or steps. A tree of stock prices is initially produced working forward from the present to expiration. At each step it is assumed that the stock price will move up or down by an amount calculated using volatility and time to expiration. This produces a binomial distribution, or recombining tree, of underlying stock prices. The tree represents all the possible paths that the stock price could take during the life of the option.

At the end of the tree – ie at expiration of the option – all the terminal option prices for each of the final possible stock prices are known as they simply equal their intrinsic values.

Next the option prices at each step of the tree are calculated working back from expiration to the present. The option prices at each step are used to derive the option prices at the next step of the tree using risk neutral valuation based on the probabilities of the stock prices moving up or down, the risk free rate and the time interval of each step. Any adjustments to stock prices (at an ex-dividend date) or option prices (as a result of early exercise of American options) are worked into the calculations at the required point in time. At the top of the tree you are left with one option price.

For example we can consider a one step binomial tree. The one step ahead stock price has two different values in a binomial model; S_0u or S_0d where u and d are the upward and downward size respectively. Then if we construct a risk neutral portfolio by shorting the derivative and buying the stock;

$$derivative : -1$$
$$share : \Delta$$

Because the portfolio is risk neutral, whether stock price goes up or down the pay off would be same, so;

$$S_0 u\Delta - f_u = S_0 d\Delta - f_d$$
$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

The cost of settin up this portfolio is;

$$S_0 - f$$

because we get a riskless position it should yield risk free rate. Then;

$$S_0 - f = e^{-rT}\left(S_0 u\Delta - f_u\right)$$

Finally option value is;

$$f = e^{-rT}\left(pf_u + (1-p)\,f_d\right)$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

To match this formulation with stock volatility, it is a popular way to define u, d and p as follows;

$$u = e^{\sigma\sqrt{\delta t}}$$
$$d = 1/u$$
$$p = \frac{e^{r\delta t} - d}{u - d}$$

The big advantage the binomial model has over the Black-Scholes model is that it can be used to accurately price American options. This is because with the binomial model it's possible to check at every point in an option's life (ie at every step of the binomial tree) for the possibility of early exercise (eg where, due to eg a dividend, or a put being deeply in the money the option price at that point is less than the its intrinsic value).

Where an early exercise point is found it is assumed that the option holder would elect to exercise, and the option price can be adjusted to equal the intrinsic value at that point. This then flows into the calculations higher up the tree and so on.

The binomial model basically solves the same equation, using a computational procedure that the Black-Scholes model solves using an analytic approach and in doing so provides opportunities along the way to check for early exercise for American options.

The main limitation of the binomial model is its relatively slow speed. It's great for half a dozen calculations at a time but even with today's fastest PCs it's not a practical solution for the calculation of thousands of prices in a few seconds.

3. Comparison of the Models

3.1. Volatility Estimation

For both models the volatility of the stock is the key factor. Estimation of the volatilty is another important topic. In this study I used three different models for Historical Average(HA), Exponentially Weighted Moving Average (EWMA) and Generalized Autoregressive Conditional Hetereoscedasitic (GARCH) model.

3.1.1. Historical Average

If we assume that conditional expectation of the volatility is constant and the daily returns has zero mean, the proper estimate of the volatility is;

$$\sigma = \sqrt{\tfrac{1}{n-1}\sum_{i=1}^{n} r^2}$$

where σ is the estimated volatility, n is the sample size and r is the daily return.

The weakest point of this model is of course constant volatility assumption. This estimate of the volatility could not mimic the big changes in the volatility and remains nearly constant where the sample size increases.

3.1.2. Exponentially Weighted Moving Average

EWMA past observations with exponentially decreasing weights to esitmate volatility. Therefore this is a modified version of historical averaging. Instead of equally weighting, in EWMA weights differ. The estimated volatility can shown as;

$$\sigma_t^2 = (1-\lambda)\, r_t^2 + \lambda \sigma_{t-1}^2$$
$$\sigma_t = \sqrt{\sigma_t^2}$$

By repeated substitutions we can re-write the forecast as;

$$\sigma_t^2 = (1-\lambda) \sum_{i=1}^{n} \lambda^{i-1} r_t^2$$

equation shows the volatility is equal to a weighting average. The weights decrease geometrically. The value of , decay factor, estimated simply by minimizing the one week forecast errors.

3.1.3. GARCH

The Generalized ARCH model of Bollerslev (1986) defined GARCH by;

$$r_t = \mu + \sigma_t \varepsilon_t$$
$$\sigma_t^2 = \lambda + \sum_{i=1}^{q} \alpha_i \left(r_{t-i} - \mu\right)^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

GARCH imposes that the proper volatility estimate is based not only on the recent volatilities and also previous forecasts which include the previous volatilities. Then the GARCH model is a long memory model. The parameters of the GARCH can be estimated by a Maximum Likelihood procedure.

3.2. Some Hypothetic Options

3.2.1. An Stock Index Option on ISE-100

First let us consider a one-month european call option with strike price 27000 (current price is taken as 25300). The price for several combinations is as follows;

	Black-Scholes	CRR
Historical Average	2207.95	2206.60
EWMA	3068.71	3071.38
GARCH	2562.63	2560.39

The CRR results obtained with 100 steps. As we can see the price of the option changes according to the volatility estimates. The volatility estimate of the historical simulation leads us to lowest price (16%). The volatility estimate of EWMA and GARCH is 30% and 22% respectively.

Now let us consider the case we have a one-month european put option with strike price 27000. The prices are;

	Black-Scholes	CRR
Historical Average	433.72	432.37
EWMA	1294.48	1297.15
GARCH	788.40	786.16

This type option price change at a higher ratio, then this option is more non-linear with respective to volatility.

For the american call option (non-divident paying) the price is same as we calculated before. But for the american put option the value will change and we can not use the Black-Scholes formula for the american put options. The table shows the prices by the binomial model;

	CRR
Historical Average	856.91
EWMA	1657.57
GARCH	1172.60

As we can see the price of the american option is higher than the european option.

3.2.2. An FX Option on USD/TRL

First let us start with a one-month european call option again with strike price 1.38 (current price is taken as 1.38). The price for several combinations is as follows;

	Black-Scholes	CRR
Historical Average	0.1419	0.1418
EWMA	0.1633	0.1631
GARCH	0.1574	0.1572

The binomail model's results obtained with 100 steps again. The volatility estimate of HA, EWMA and GARCH is 14% , 22 and 20% respectively.

Another case, we have a one-month european put option with strike price 1.38. The prices are;

	Black-Scholes	CRR
Historical Average	0.0106	0.0104
EWMA	0.0320	0.0318
GARCH	0.0261	0.0259

The table shows the prices by the binomial model result for the american version of the last option;

	CRR
Historical Average	0.0220
EWMA	0.0458
GARCH	0.0395

As we can see the price of the american option is higher than the european option again.

4. Conclusion

Due to its nonlienar pay offs pricing an option is quite difficult with respective to other financial instruments like fixed income instruments. One should consider the possible future oturcomes of the underlying asset. If the option is an exotic that is effected by more than one sources of the uncertainty it becomes more complicated. In this paper we just introduce an compare two popular models.

The results show that the prices of the two models quite similar where the number of steps in the binomial. Since Black-Scholes model can not calculate the price of the american option one can use the binomial model with high number steps and can similar result as if Black-Scholes.

Another result shows that due to high volatility the option price on the ISE will be very high. For example a long position on one-month american call with strike price 27000 might have positive payoff if the prices goes somewhere around 30000. On the other hand the price of the FX options is more reasonable.

By the results of the paper we can see that the volatility estimation is also as important as option pricing. Because the volatilty changes the price might change exponentially.